ON THE VERTICAL CIRCULATION IN FRONTAL ZONES

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Summary: The quasi-geostrophic theory is applied to study the transverse circulations in fronts caused by variations in geostrophic wind and temperature along the frontal zone. Previous results of SAWYER are confirmed. The decisive quantity seems to be the Jacobian determinant of the front-parallel and front-normal components of the geostrophic wind, in a vertical cross-section normal to the front. It is found that the transverse motions will circulate around those regions in the cross-section where the Jacobian mentioned above is large; the circulation cells will tend to slope along the vector lines of the absolute vorticity.

1. Introduction. The familiar concept of atmospheric fronts has a peculiar dual character. On the one hand, a front may be described as a sloping shallow layer of transition between two horizontal air currents of different temperature, which balance each other in accordance with the formula of MARGULES. This picture may be extended to include also the associated jet stream and tropopause structure.

Fronts have also other attributes, however: cloud systems and precipitation patterns, which must be due to organized ascending motions along the frontal slope. More recently, it has been observed that also pronounced descending motions, as revealed by remarkably low relative humidities, take place within frontal zones. (VUORELA 1953, SAWYER 1955).

It is probably fair to say that the physical link between the wind and temperature fields on the one hand, and the rainproducing vertical circulation on the other, is not too well understood.

In this connection it is interesting to note that the starting point for the discoveries of the Bergen school was the study of the *lines of convergence* in the horizontal wind field and the associated vertical motions. Already during his stay in Leipzig 1913—17, VILHELM BJERKNES initiated a careful investigation of these convergence lines (see V. BJERKNES (1933)) and the work was continued in Bergen in 1917 by means of a close network of observing stations along the Norwegian west coast, which V. BJERKNES had organized for this purpose. It was through these studies of convergence lines that the polar front cyclone was discovered by V. BJERKNES' collaborators. In his famous paper "On the structure of moving cyclones", J. BJERKNES (1919) describes the fronts as convergence lines and also discusses the associated vertical motions. Thus the vertical circulations were stressed right from the beginning of frontal meteorology.

Later, BERGERON (1928) pointed out that bands of strong isotherm concentration are formed by differential advection in non-divergent horizontal deformation fields. This process undoubtedly accounts for the initial formation of most frontal zones. It is well to remember, however, that horizontal, non-divergent motions cannot produce the frontal wind shear or the associated jet streams; the formation of these features requires vorticity generation, which can only be brought about by convergent or divergent winds and vertical motions, i.e. by vertical circulations.

In a relatively uniform frontal zone, where gradients of wind and temperature along the zone are weak, the vertical circulations may be assumed to occur as transverse circulations in vertical planes normal to the zone. Such circulations are highly effective in changing the distributions of temperature and front-parallel wind; the rate of change of these quantities may readily be calculated from the dynamic and thermodynamic equations if the transverse circulations are known.

Vice versa, if the changes in temperature and front-parallel winds are observed, conclusions may be drawn concerning the transverse circulation that caused them. This principle was used by J. BJERKNES (1950) and, in a different way by NEWTON (1954) to obtain a picture of the transverse motion in special synoptic situations.

The vertical circulations in fronts may also be determined from the quasi-geostrophic theory, i.e. by requiring that the wind and temperature changes brought about by the combined effect of (1) geostrophic advection of vorticity and temperature and (2) the vertical motions and the associated divergent winds, shall not disturb the geostrophic and hydrostatic balance. From this principle, the vertical circulations *and* the local changes in wind and temperature can be calculated when the instantaneous (geostrophic) wind and temperature distributions are known. It is thus not necessary to know the local changes in order to determine the vertical circulations.

The applicability of the quasi-geostrophic theory to fronts might be questioned on the basis of the small scale of the frontal structures and the enormous local and convective accelerations occurring in moving fronts. To this it may be answered that the smallnes of scale in the direction across the front is of no consequence since the air particles do not move across it, but follow trajectories more or less parallel with the front. The use of the geostrophic approximation requires only smallness of the particle accelerations, and these do not appear to be excessive in frontal zones. Moreover, the transverse circulation systems, as revealed by the areas of continuous precipitation, have a time scale of several days, indicating that the phenomenon cannot be explained in terms of gravity-inertia oscillations.

According to the quasi-geostrophic theory, the principal effect which controls vertical motions is the *advection of vorticity by the thermal wind* (in accordance with SUTCLIFFE's (1947) "development" principle). This immediately explains why frontal zones with their strong thermal winds are preferred locations for ascending and descending air currents; whereas in nearly barotropic regions, vertical motions (apart from small-scale convection) are likely to be slight.

Moreover, it follows from the quasi-geostrophic theory that whenever the horizontal temperature gradient is increased as a result of differential advection in a horizontal deformation field, a vertical circulation in the thermodynamically direct sense must be set up, in order to produce the corresponding increase in the vertical wind shear. This mechanism has been invoked by NAMIAS and CLAPP (1949) to explain the formation of jet streams. SAWYER (1956) has shown how the resulting vertical circulation can be calculated on the basis of the quasi-geostrophic theory. In an earlier paper, SAWYER (1952) found that a direct circulation will also be produced if the thermal wind has a component across the front directed from the cold towards the warm air. This result was also obtained from the quasi-geostrophic theory, however, in a more primitive form.

In a previous article by the author (1959) the quasi-geostrophic theory was used to study those transverse circulations which are set up in a frontal zone by the combined effect of friction in the Ekman layer and heat of condensation. The result seems to indicate that these effects give an irreversible mechanism which tends to change diffuse fronts into sharp fronts near the earth's surface.

The present study, which perhaps may be considered as an extention of SAWYER'S (1956) paper, deals with transverse circulations set up in response of variations in wind and temperature along the frontal zone.

2. Derivation of the differential equation for transverse circulation in frontal zones. The following considerations will be restricted to frontal zones which are sufficiently straight so that their curvature may be ignored. We choose a Cartesian coordinate system with the x-axis along the zone and the y-axis pointing towards colder air. Using pressure p as vertical coordinate, the components U, V of the geostrophic wind are

$$U = -\frac{1}{f}\frac{\partial \Phi}{\partial y}, \qquad V = \frac{1}{f}\frac{\partial \Phi}{\partial x} \tag{1}$$

where Φ denotes geopotential and f the Coriolis parameter. Since frontal zones are relatively small-scale features, the Coriolis parameter will be considered constant, and we have

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{2}$$

The hydrostatic equation may be written

$$\frac{1}{f}\frac{\partial \Phi}{\partial p} = -\frac{\alpha}{f} = -\gamma(p)\theta \tag{3}$$

where α denotes specific volume, θ potential temperature, and

$$\gamma(p) = \frac{R}{fp_0}\left(\frac{p_0}{p}\right)^{c_v/c_p}, \qquad p_0 = 1000 \text{ mb} \tag{4}$$

with R denoting gas constant and c_v and c_p specific heats.

From (1) and (3) we obtain the termal wind equations

$$\frac{\partial U}{\partial p} = \gamma\frac{\partial \theta}{\partial y}, \qquad \frac{\partial V}{\partial p} = -\gamma\frac{\partial \theta}{\partial x} \tag{5}$$

Let u, v denote the components of the departure from geostrophic wind and ω the individual rate of change of pressure. The equation of motion in the direction of x then is

$$\frac{DU}{dt} + \frac{Du}{dt} = fv \tag{6}$$

where

$$\frac{D}{dt} = \frac{\partial}{\partial t} + (U + u)\frac{\partial}{\partial x} + (V + v)\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \tag{7}$$

denotes the individual derivative.

We may assume that the motion is nearly geostrophic, so that u is at all times small compared with characteristic values of $|U|$. This means that $\frac{Du}{dt}$ must either be small, or must be oscillatory, since significant progressive changes in u cannot occur. As we want to eliminate gravity-inertial oscillations and consider the quasi-balanced motion, we may therefore ignore $\frac{Du}{dt}$ in (6), and write the equation of motion in the form

$$\frac{DU}{dt} = fv \tag{8}$$

The use of this approximation means to keep the non-geostrophic advection terms $u\partial U/\partial x + v\partial U/\partial y + \omega\partial U/\partial p$, whereas the terms $U\partial u/\partial x + V\partial u/\partial y$ are ignored. Since these terms are seemingly of the same order of magnitude, it might be questioned why we do not drop them all, and write instead of (8),

$$\frac{D_g U}{dt} = fv \tag{9}$$

where

$$\frac{D_g}{dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} \tag{10}$$

is the individual derivative in geostrophic motion.

To justify this, we integrate the left-hand sides of (6), (8), and (9) with respect to time between the limits t and $t + \tau$, and compare the results. To this purpose we consider a particle which at time t is located in the point x, y, p. The particle moves with the speed $U + u$, $V + v$, ω and arrives at the time $t + \tau$ in the point $x + \xi$, $y + \eta$, $p + \pi$. If the particle had moved only with the geostrophic wind, it would instead have ended up at the time $t + \tau$ in the point $x + \xi_g$, $y + \eta_g$, p. Subtracting the left-hand sides of (6) and (8), and integrating, we find

$$\int_t^{t+\tau} \left(\frac{D(U+u)}{dt} - \frac{DU}{dt} \right) dt = \int_t^{t+\tau} \frac{Du}{dt} dt = u(x+\xi, y+\eta, p+\pi, t+\tau) - u(x,y,p,t) \quad (11)$$

which is a small quantity for all τ, since u is assumed to be small everywhere.

On the other hand, we get by subtracting the left-hand sides of (6) and (9) and integrating,

$$\int_t^{t+\tau} \left(\frac{D(U+u)}{dt} - \frac{D_g U}{dt} \right) dt = u(x+\xi, y+\eta, p+\pi, t+\tau) - u(x,y,p,t) \quad (12)$$
$$+ U(x+\xi, y+\eta, p+\pi, t+\tau) - U(x+\xi_g, y+\eta_g, p, t+\tau)$$

This expression is not bounded by the assumption of approximate geostrophic balance; it will increase progressively with τ and may soon become significant, in particular in frontal zones where the gradient of U is large. It is thus seen that the terms $u\partial U/\partial x + v\partial U/\partial y + \omega\partial U/\partial p$, may, in contrast to $U\partial u/\partial x + V\partial u/\partial y$, represent significant progressive velocity changes. For this reason, it is believed that the former terms should be kept, i.e. that the geostrophic approximation should be used in the form (8), rather than the form (9).

With the notation (10), eq. (8) may be written

$$\frac{D_g U}{dt} + u\frac{\partial U}{\partial x} + v\frac{\partial U}{\partial y} + \omega\frac{\partial U}{\partial p} = fv \quad (13)$$

Likewise, the thermodynamic energy equation may be written:

$$\frac{D_g \theta}{dt} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + \omega\frac{\partial \theta}{\partial p} = \frac{D\theta}{dt} \quad (14)$$

where $\frac{D\theta}{dt}$ represents a heat source.

A frontal zone is characterized by large lateral and vertical gradients of wind and temperature, whereas the longitudinal gradients are weak. We may therefore in (13) and (14) ignore $u\partial/\partial x$ compared with $v\partial/\partial y$ and $\omega\partial/\partial p$. Furthermore, we introduce the "absolute momentum"

$$m = U - fy \quad (15)$$

Eqs. (13) and (14) then become

$$\frac{D_g U}{dt} + v\frac{\partial m}{\partial y} + \omega\frac{\partial m}{\partial p} = 0 \tag{16}$$

$$\frac{D_g \theta}{dt} + v\frac{\partial \theta}{\partial y} + \omega\frac{\partial \theta}{\partial p} = \frac{D\theta}{dt} \tag{17}$$

We now apply the operator $\partial/\partial p$ to (16) and $\gamma\partial/\partial y$ to (17). Recalling the definition (10), and using (3) and (5), we obtain

$$\frac{D_g}{dt}\frac{\partial U}{\partial p} = -\frac{\partial U}{\partial p}\frac{\partial U}{\partial x} - \frac{\partial V}{\partial p}\frac{\partial U}{\partial y} - \frac{\partial}{\partial p}\left(v\frac{\partial m}{\partial y} + \omega\frac{\partial m}{\partial p}\right) \tag{18}$$

$$\frac{D_g}{dt}\gamma\frac{\partial \theta}{\partial y} = \frac{\partial U}{\partial p}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial p}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\gamma\left(v\frac{\partial \theta}{\partial y} + \omega\frac{\partial \theta}{\partial p}\right) + \gamma\frac{\partial}{\partial y}\frac{D\theta}{\partial t} \tag{19}$$

On account of the first equation (5), it follows that the right-hand sides of (18) and (19) must be equal. The contribution to the right-hand sides of (18) and (19) from purely geostrophic motion is in both equations measured by the expression $\frac{\partial \boldsymbol{V}}{\partial p} \cdot \nabla U$, but appears with opposite sign in the two equations. Therefore these contributions can never be equal except when they are both zero. Moreover, the differential heating term $-\gamma\frac{\partial}{\partial y}\frac{D\theta}{dt}$ appears only in (19). It follows that whenever $\frac{\partial \boldsymbol{V}}{\partial p} \cdot \nabla U$ is different from zero, or when differential heating occurs, non-geostrophic motions must take place in order to make the right-hand sides of (18) and (19) become equal.

Equating the right-hand sides of (18) and (19) we obtain

$$\begin{aligned} -\frac{\partial}{\partial y}\left(v\,\gamma\frac{\partial \theta}{\partial y} + \omega\gamma\frac{\partial \theta}{\partial p}\right) + \frac{\partial}{\partial p}\left(v\frac{\partial m}{\partial y} + \omega\frac{\partial m}{\partial p}\right) \\ = -2\frac{\partial U}{\partial p}\frac{\partial U}{\partial x} - 2\frac{\partial V}{\partial p}\frac{\partial U}{\partial y} - \gamma\frac{\partial}{\partial y}\frac{D\theta}{dt} \end{aligned} \tag{20}$$

On account of (2), the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial p} = 0 \tag{21}$$

Here we ignore the first term, with the justification that the derivatives taken along the frontal zone are small compared with the derivatives across it. We may therefore introduce a stream function ψ for the non-geostrophic motion in the yp-plane:

$$v = -\frac{\partial \psi}{\partial p}, \qquad \omega = \frac{\partial \psi}{\partial y} \tag{22}$$

When these expressions are inserted into (20) we obtain a self-adjoint equation in the single variable ψ:

$$-\frac{\partial}{\partial y}\left(\gamma \frac{\partial \theta}{\partial p}\frac{\partial \psi}{\partial y} - \gamma \frac{\partial \theta}{\partial y}\frac{\partial \psi}{\partial p}\right) + \frac{\partial}{\partial p}\left(\frac{\partial m}{\partial p}\frac{\partial \psi}{\partial y} - \frac{\partial m}{\partial y}\frac{\partial \psi}{\partial p}\right) = Q \tag{23}$$

where the forcing term Q has the form:

$$Q = -2\frac{\partial U}{\partial p}\frac{\partial U}{\partial x} - 2\frac{\partial V}{\partial p}\frac{\partial U}{\partial y} - \gamma \frac{\partial}{\partial y}\frac{D\theta}{dt} \tag{24}$$

The coefficients of equation (23) have the following meaning:

$-\gamma \frac{\partial \theta}{\partial p} > 0$ measures static stability

$-\gamma \frac{\partial \theta}{\partial y} = -\frac{\partial m}{\partial p} > 0$ measures baroclinicity

$-\frac{\partial m}{\partial y} = f - \frac{\partial U}{\partial y} > 0$ is the absolute vorticity.

These three quantities are all assumed positive.

Moreover, the frontal layer will be assumed to be stable with respect to transverse overturnings. From the theory of the stability of baroclinic currents and vortices, it then follows that

$$\delta = \gamma \left(\frac{\partial \theta}{\partial p}\frac{\partial m}{\partial y} - \frac{\partial \theta}{\partial y}\frac{\partial m}{\partial p}\right) > 0 \tag{25}$$

showing that eq. (23) is of the elliptic type. The solution is therefore uniquely defined if the frontal zone is assumed to be inside a closed boundary curve in the yp-plane, where the boundary condition

$$\psi = 0 \tag{26}$$

applies. The boundary curve may be taken as the top of the atmosphere and the earth's surface, connected by two lateral walls at $y = \pm \infty$.

It is thus seen that the determination of ψ is mathematically equivalent to determining the electrostatic potential distribution in non-homogeneous and anisotropic dielectricum between two conducting parallel plates kept at the same potential, for a given distribution of charges.

Eq. (23) was applied to fronts by SAWYER (1955) (with only the first term appearing on the right-hand side) and by the writer (1959) (with only the last term appearing on the right). The same kind of equation holds for forced meridional circulations in a baroclinic vortex (A. ELIASSEN 1951, eq. (29)).

3. Simplification of (23) by change of coordinates. The left-hand side of (23) will assume a simpler form in an oblique system of coordinates, by replacing the vertical coordinate lines by the slanting lines along the absolute vorticity vector. This may be done by introducing m and p instead of y and p as independent variables. Denoting by a subscript the quantity which is held constant, we have the transformation formulae:

$$\left(\frac{\partial}{\partial y}\right)_p = \frac{1}{\left(\dfrac{\partial y}{\partial m}\right)_p}\left(\frac{\partial}{\partial m}\right)_p$$
$$\left(\frac{\partial}{\partial p}\right)_y = \left(\frac{\partial}{\partial p}\right)_m - \frac{\left(\dfrac{\partial y}{\partial p}\right)_m}{\left(\dfrac{\partial y}{\partial m}\right)_p}\left(\frac{\partial}{\partial m}\right)_p \tag{27}$$

Here the operators on the left are the same ones as those occuring in (23). The operator $\left(\frac{\partial}{\partial p}\right)_m$ means differentiation along the slanting lines $m =$ constant, whose tangent is the absolute vorticity vector. The expression $\left(\frac{\partial y}{\partial p}\right)_m$ characterizes the slope of these lines; from the thermal wind equation (5) it follows that

$$-\left(\frac{\partial y}{\partial p}\right)_m = \gamma\left(\frac{\partial \theta}{\partial m}\right)_p > 0 \tag{28}$$

showing that the lines $m =$ constant tilt towards colder air with height. Furthermore, we note that the expression

$$-\left(\frac{\partial y}{\partial m}\right)_p = \left(f - \frac{\partial U}{\partial y}\right)^{-1} > 0 \tag{29}$$

is the inverse of the absolute vertical vorticity.

It follows from (27) that Jacobian determinants are transformed in the following way:

$$\frac{\partial(\alpha, \beta)}{\partial(y, p)} = \frac{1}{\left(\dfrac{\partial y}{\partial m}\right)_p}\frac{\partial(\alpha, \beta)}{\partial(m, p)} \tag{30}$$

In particular, the discriminant (25) assumes the form

$$\delta = \frac{\gamma\left(\dfrac{\partial \theta}{\partial p}\right)_m}{\left(\dfrac{\partial y}{\partial m}\right)_p} = -\gamma\left(\frac{\partial \theta}{\partial p}\right)_m\left(f - \frac{\partial U}{\partial y}\right) \tag{31}$$

By applying the transformation formulae (27) and (30) to the left-hand side of (23) (but not to the right-hand side), and making use of (28), (29), and (31), we find

$$\left[\frac{\partial}{\partial m}\left(\delta\frac{\partial\psi}{\partial m}\right)\right]_p + \left(\frac{\partial^2\psi}{\partial p^2}\right)_m = \frac{Q}{f-\dfrac{\partial U}{\partial y}} \tag{32}$$

This is the form of the differential equation for ψ in the coordinates m and p. In contrast to (23), the left-hand side of (32) contains no mixed derivatives, and has only one variable coefficient δ.

When the distributions of U, θ and Q in the yp-plane are known, ψ may be determined by numerical solution of the differential equation, in either of the forms (23) or (32). The latter form seems to be preferable, partly because it is simpler, and also because it gives a better resolution of the frontal zone where wind gradients tend to be strong.

Such numerical solutions will not be given here; instead we shall attempt to draw some conclusions regarding the general character of the solution, based on qualitative reasoning.

Since $\psi = 0$ along the boundary, the streamlines cannot cross the boundary, but must close up on themselves. By integrating (32) over the area bounded by a closed streamline without stagnation points, it can be shown that if the circulation along the streamline is in the clockwise sense (viewed from the positive x-axis), then the total "source strength" $\iint Q\,dy\,dp$ inside the streamline must be positive, and vice versa (provided $\delta > 0$ everywhere). It follows that there can be no closed streamlines in a region where $Q = 0$. The air will circulate around those areas where the largest (positive or negative) values of Q are located.

Some further information concerning the form of the streamlines may be obtained by considering the Green's function, i.e. the solution when Q is zero everywhere except in a point source. The Green's function of eq. (32) will, in the vicinity of a point source in the interior, have the form of a logarithmic potential. The corresponding streamlines are concentric ellipses in the mp-plane, encircling the point source; their principal axes are oriented in the directions p = constant and m = constant, the ratio of their semi-axis $\triangle m$ and $\triangle p$ being $\triangle m/\triangle p = \delta^{\frac{1}{2}}$ (Fig. 1a).

By transforming this streamline pattern in the mp-plane back to the yp-plane we obtain, apart from a constant factor, the Green's function of eq. (23). In the vicinity of the point source, the streamlines in the yp-plane are again concentric ellipses, but their principal axes are tilted; their shape and orientation are most easily inferred from the fact that the diameters p = constant and m = constant must retain their property of being conjugate diameters (Fig. 1b). A positive point source gives a circulation in the clockwise sense, and vice versa.

This elliptical streamline pattern will be found only in the vicinity of the point source; at greater distances, the streamlines must change shape so as to adjust to the boundaries.

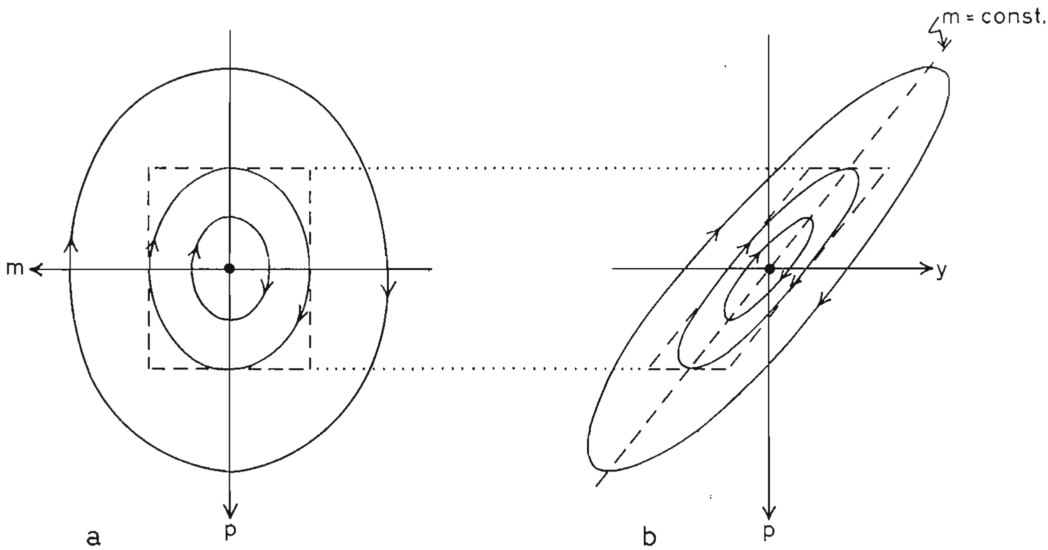

Fig. 1. Streamlines around a positive point source a. in the mp-plane, b. in the yp-plane.

Distortions of the elliptical streamlines in the mp-plane at some distance from the point source will also be produced by variations in the coefficient δ. Just as in the corresponding two-dimensional electrostatic problem the equi-potential lines will tend to avoid regions of high dielectric constant, the streamlines will tend to spread out in regions where δ is large, and to become crowded where δ is small. It should be noted, however, that the variations of δ represent an anisotropic effect, which has no significance where $\partial\psi/\partial m = 0$, i.e. where the streamlines are horizontal. In actual atmospheric cross-sections, there are usually very considerable variations in δ; thus there is normally a region of very small δ just south of the jet stream, and besides, δ increases systematically with height and tends to infinity as $p \to 0$. Clearly these variations must have a strong effect upon the form of the Green's function of eq. (32), and these distortions must also be reflected in the form of the corresponding streamline pattern in the yp-plane. However, the solution must still have the character of a circulation around the source region.

Since the circulations around different sources are additive, we may from the properties of the Green's function derive qualitatively the principal features of the circulation produced by an arbitrary distribution of sources, if it is not too complicated.

4. Qualitative discussion of transverse circulations in the dry-adiabatic case. We shall in this section ignore the heating in eq. (24). By means of the thermal wind equations (5), the source density Q may then be written

$$Q = -2\gamma\left(\frac{\partial U}{\partial x}\frac{\partial \theta}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial \theta}{\partial x}\right) \tag{33}$$

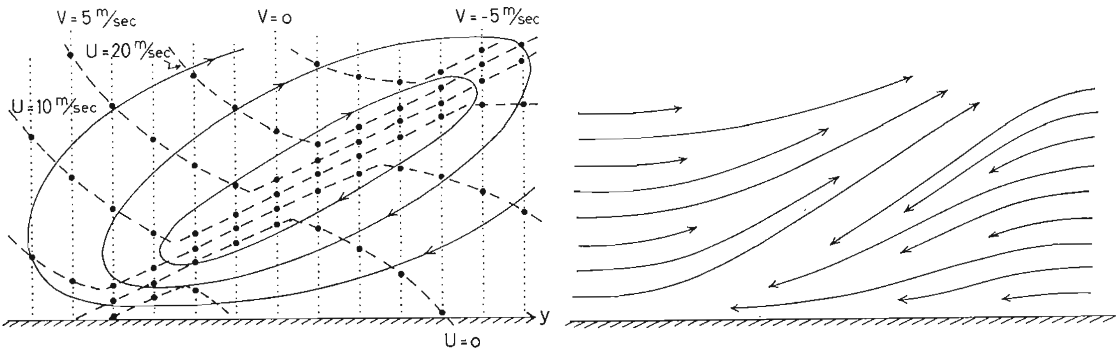

Fig. 2. Transverse motion in an idealized frontal zone where $\partial V/\partial y < 0$, $\partial V/\partial p = 0$.

a. Dashed lines: U-isotachs. Dotted lines: V-isotachs. Solid lines: Streamlines of transverse non-geostrophic circulation.

b. Streamlines of convergent total (geostrophic and non-geostrophic) transverse motion relative to the motion of the front.

The distribution of this expression in an isobaric surface may be determined graphically from the density of intersection points between the two families of curves $U =$ constant and $\theta =$ constant. If the isotherms are given a positive direction such that warmer air is to the right, it will be seen that Q is positive where the isotherms run towards larger values of U.

For the sake of discussion, we shall write (33) in the form

$$Q = 2\left(\frac{\partial U}{\partial p}\frac{\partial V}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial V}{\partial p}\right) \tag{34}$$

which is obtained by applying (5) and (2). It is seen from this expression that the distribution of Q in the vertical yp-plane is obtained from the density of intersection points between the curves $U =$ constant and $V =$ constant in this plane. Recalling that a positive source ($Q > 0$) will produce a circulation in this plane in the clockwise sense (viewed from the positive x-axis), it follows that the circulation around a source will take place in the direction given by a rotation from ∇V towards ∇U in the yp-plane (the fact that y and p have different dimensions is of no consequence in this connection). From these rules, one may immediately get a qualitative picture of the transverse circulation, once the curves $U =$ constant and $V =$ constant in the yp-plane are known.

We shall apply this qualitative method to a schematic frontal zone in the lower troposphere, between two slightly baroclinic air masses. The assumed distribution of U in the yp-plane is shown by the dashed isotachs in Fig. 2a. Assuming furthermore a frontogenetic horizontal deformation field such that $\partial V/\partial y < 0$ and $\partial V/\partial p = 0$ (i.e. all isotherms in isobaric surfaces are assumed parallel with the front), the distribution of V is as shown by the dotted lines in Fig. 2a. This corresponds to the case treated by Sawyer (1956). The source therm Q, as revealed by the density of intersection points between these two families of curves, has a large positive value in the frontal layer, and small positive values in the air masses on both sides. The resulting transverse

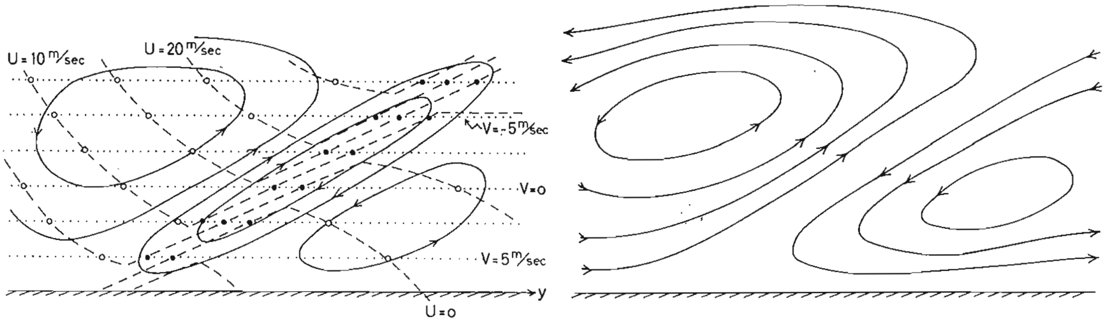

Fig. 3. Transverse motion in an idealized frontal zone where $\partial V/\partial y = 0$, $\partial V/\partial p > 0$.

a. Dashed lines: U-isotachs. Dotted lines: V-isotachs. Solid lines: Streamlines of transverse non-geostrophic circulation.

b. Streamlines of total (geostrophic and non-geostrophic) transverse motion, relative to the motion of the front.

circulation must be a direct one around the frontal layer, as indicated by the streamlines (solid) in Fig. 2a. This circulation will counteract the effect of the geostrophic wind field which tends to steepen the frontal slope.

It was shown in section 3 that the circulation cell produced by a single point source will slope along the lines m = constant (Fig. 1b); but these lines are usually found to run nearly parallel with the frontal layer. It follows that the transverse circulation cell produced by a distribution of sources in the frontal layer must tilt along the front itself, as shown in the figure. The vertical motion will thus have the character of an up- or down-gliding motion along the frontal slope.

Fig. 2b shows the streamlines of the convergent total transverse field of motion $(V + v, \omega)$.

Next let us consider a different distribution of V, defined by $\partial V/\partial y = 0$, $\partial V/\partial p > 0$, so that there is a certain component of the thermal wind across the front, from the cold towards the warm air. This is typical of many cold fronts, where $\partial \theta/\partial x < 0$. Assuming the same distribution of U as before, the isotachs of U and V will run as shown in Fig. 3a.

In this case it is seen that Q is small (slightly negative) in both air masses and positive in the frontal layer. There will thus be a direct circulation around the frontal zone, as shown by the streamlines (solid) in Fig. 3a. As in the former case, this circulation will counteract the tendency of the geostrophic (thermal) wind to steepen the front.

If the geostrophic wind V (relative to a reference system which travels with the front) is added to the non-geostrophic circulation (v, ω), we obtain the total transverse motion, shown by the streamlines in Fig. 3b. This picture is in good agreement with the distribution of clouds and humidity in many cold fronts.

If the gradient of V is reversed, i.e. $\partial V/\partial p < 0$ or $\partial \theta/\partial x > 0$, as will often be the case in warm fronts, we find $Q < 0$ in the frontal zone. The corresponding transverse circulation will be an indirect one.

In similar manner, we might investigate the transverse circulation for other distributions of V. In particular, no circulation will be set up if the V-isotachs are parallel with the U-isotachs in the frontal layer.

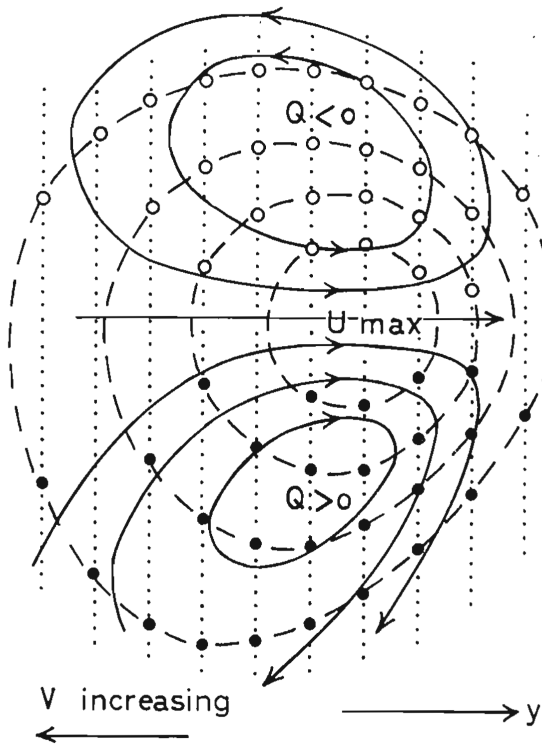

Fig. 4. Transverse circulation in the vicinity of an idealized jet core when $\partial V/\partial y < 0$, $\partial V/\partial p = 0$. Dashed lines: U-isotachs. Dotted lines: V-isotachs. Solid lines: Streamlines of transverse (non-geostrophic circulation.

It is noteworthy that the total source strength of the frontal layer, represented by the total number of intersection points between the U-lines and the V-lines, in the cases considered is independent of the thickness of this layer, as long as the frontal tangential wind contrast $\triangle U$ is kept constant. The transverse circulation therefore does not depend critically on the thickness of the frontal layer, but only on the tangential wind contrast (and the distribution of V).

The method may also be applied to the transverse circulation in jet streams, where the isotachs of U form closed curves in the yp-plane. We note that the total source strength inside such a closed isotach must vanish. This may be seen by integrating (34) over a region σ bounded by a closed countor S:

$$\int_{\sigma} Q\, dy\, dp = -2 \oint_{s} U dV \tag{35}$$

where the integration along S is taken in the clockwise sense (viewed from the positive x-axis). If U is constant along S, the right-hand side is seen to vanish.

As a consequence, the distribution of Q in the jet stream will usually have the character of a doublet, and the resulting transverse circulation will take the form of two circulation cells with opposite rotation on both sides of the jet stream, and between them a current right across the jet stream core.

As an example, we consider a distribution of U as shown by the dashed isotachs in Fig. 4, and a confluent field of V as shown by the dotted isotachs. It follows that Q is positive below the jet stream core and negative above it, so that the resulting transverse circulation must form two direct circulation cells as shown by the solid lines in the diagram. In this case we must take a certain reservation regarding the effect upon the streamlines of the region of small δ-values on the warm side of the jet stream. This effect can hardly be estimated without solving (32) numerically.

5. Concluding remarks. The kinematic source term (34) considered in the preceding section constitutes only a part (but probably a major part) of the total forcing term which governs the transverse circulation in fronts. Another forcing effect is due to influx from the Ekman layer, and still another results from differential diabatic heating. The resultant circulation produced by these various effects will be enhanced by the released heat of condensation in the ascending currents.

Even if all these effects were taken into account, we cannot expect to obtain a complete description of the transverse circulation in fronts from a two-dimensional theory. In real fronts there are considerable variations of the kinematic structure from one cross-section to another; in particular, we find that the kinematic source term (34) varies considerably along the front. Apparently the gradients along the front are not always sufficiently small to justify their neglect compared with the gradients across the front. Therefore, an accurate quantitative treatment of the transverse circulations in fronts must be based on a full three-dimensional theory.

However, the two-dimensional theory enables us to demonstrate, in a relatively simple manner, how the transverse circulations in fronst are linked with the horizontal, geostrophic flow, and may thus perhaps represent a mental aid for our understanding of the dynamics of fronts.

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